Pseudo-Permutations I: First Combinatorial and Lattice Properties

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Abstract

In this paper, we introduce new combinatorial objects, the pseudo-permutations, which are a generalization of the permutations. Pseudo-permutations naturally appear in various fields of Computer Science and Mathematics. We provide the first combinatorial results on these objects: we study the classical statistics of enumeration, inversions, descents and we prove that the set of all the pseudo-permutations is a lattice.
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1 Introduction

In various fields of Computer Science (Artificial Intelligence, Knowledge representation, ...), one has to deal with temporal knowledge: one considers a set of events which happen at certain dates, and wants to use this information to solve a problem, take a decision. However, it is often not meaningful *when* the events occur, while the relevant information is the *temporal relations* between events: did event i happened before, during or after event j ? In this context, it is natural to represent the temporal relations between n events by an

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ordered sequence of nonempty parts of $[\![1,n]\!]$ such that each integer appears exactly once. If i is in a part of $[\![1,n]\!]$ which appears in the sequence before the part which contains j, then the event i happened before the event j. If they appear in the same part, they occured simultaneously. For example, the sequence $\{1\}\{3,4\}\{2\}$ means that event 1 occured first, event 2 occured last, and events 3 and 4 occured at the same time. In the following, we will call such a sequence a *pseudo-permutation of order* n (in the example, $n=4$), we will use parenthesis instead of braces, and we will remove unnecessary comas. Therefore, we will write the example as (1)(34)(2).

Having defined the pseudo-permutations, one can study their combinatorial properties. They appear to have a very rich structure and many interesting properties. Among these, one can see that their enumeration is related to Eulerian numbers, and that an inversion table associated with each pseudo-permutation can be defined. This induces a partial order on pseudo-permutations compatible with the inversion tables and we will see that this order is a lattice, as it is the case for the usual permutations. One can also define the descents of a pseudo-permutation and prove that this notion has the same properties as the usual descents on permutations. About the lattice structure, one can also see that the set can be divided in connected components that are hypercubes, the dimensions of which can naturally be interpretated in combinatorial terms. The aim of this paper is to show how rich can be this structure and to give a first insight in it, given that a second paper is in preparation to provide new interesting properties about pseudo-permutations. Since this paper is intented to present this new object, we will only give the sketches of the proofs and will publish a long version elsewhere.

The paper is structured as follows. We first recall the definitions of a lattice and the usual properties of Eulerian numbers (Section 2). We then concentrate on the pseudo-permutations themselves and define the pseudo-permutohedron (Section 3). Next, we prove their first combinatorial properties (Section 4) and their first lattice properties (Section 5).

2 Preliminaries

We recall here a few definitions and basic results used in the rest of the paper. We will also use the following standard notations. We will denote by $[\![i,j]\!]$ the set $\{k \in \mathbb{N} \mid i \leq k \leq j\}$ and by $\mathfrak{S}_n$ the set of all the permutations of order n, i.e. the set of all the sequences of n integers which contain each integer in $[\![1,n]\!]$ exactly once. The *permutohedron* of order n is then the directed graph defined over $\mathfrak{S}_n$ by: there is an edge from σ to σ' if σ' is obtained from σ by switching two neighbour integers i and j in σ such that $i < j$.

2.1 Descents and Eulerian numbers

The Eulerian numbers are very classical numbers and the reader can refer to [FS70, Com70] for a complete view about them. In this paper, we will only recall their usual definitions and properties.

Let σ be a permutation. One says that σ has a *descent* in i, or equivalently that i is a *descent* of σ, if $\sigma(i) > \sigma(i+1)$. The *descent number* of σ is then the number of descents of σ. The *descent set* of σ is the set of the descents of σ. If one denotes by $a_{n,k}$ the number of elements of $\mathfrak{S}_n$ with k descents, one has the following formulas:

$$a_{n,k} = a_{n,n-k-1} = (n-k)a_{n-1,k-1} + (k+1)a_{n-1,k},$$

with $n > 1$ and $k > 1$. Moreover, $a_{n,0} = 1$ and $a_{0,k} = 0$ for all n and k. The *Eulerian numbers* are very close to these numbers since they can be defined by $A_{n,k} = a_{n,k-1}$. Moreover, the generating series of these numbers is known and one has:

$$1 + \sum_{1 \leq k \leq n} a_{n,k-1} \frac{t^n}{n} u^{k-1} = \frac{1-u}{e^{t(u-1)} - u}.$$

One can define in the same way the rises of a permutation and the rise number of a permutation.

2.2 Lattice theory

We recall that an *order relation* is a binary relation $\leq$ over a set, such that for all x, y and z in this set, $x \leq x$ (reflexivity), $x \leq y$ and $y \leq z$ implies $x \leq z$ (transitivity), and $x \leq y$ and $y \leq x$ implies $x = y$ (antisymmetry). Such a relation is often called a *partial order*. The set is then a *partially ordered set* or, for short, a *poset*.

A *lattice* is a poset such that any two elements a and b have a least upper bound (called *supremum* of a and b and denoted by $a \vee b$) and a greatest lower bound (called *infimum* of a and b and denoted by $a \wedge b$). The element $a \vee b$ is the smallest element among the elements greater than both a and b. The element $a \wedge b$ is defined dually. Lattices are strongly structured sets, and many general results are known about them. For example, efficient coding and algorithms are known for lattices. For more details, see for example [DP90].

3 Pseudo-permutations

3.1 Definitions

Let n be an integer. The set $\mathfrak{P}(n)$ of the pseudo-permutations of order n is the set of sequences of non-empty parentheses such that each integer in

$[\![1, n]\!]$ appears exactly once. In other words, the set $\mathfrak{P}(n)$ is the set of ordered partitions[1] of $[\![1, n]\!]$ with nonempty parts.
For example, here is the complete set of $\mathfrak{P}(3)$:

$$\mathfrak{P}(3) = \{ \quad (1)(2)(3),\ (1)(3)(2),\ (2)(1)(3),\ (2)(3)(1),\ (3)(1)(2),\ (3)(2)(1), \\ (1)(23),\ (2)(13),\ (3)(12),\ (23)(1),\ (13)(2),\ (12)(3),\ (123) \quad \}$$

Since the order inside the parentheses is irrelevant, we will generally write the integers in the parentheses in increasing order. Notice also that there exists an obvious embedding from the permutations into the pseudo-permutations that maps a permutation $\sigma = (\sigma_1, \ldots, \sigma_n)$ to $(\sigma_1)(\sigma_2)\cdots(\sigma_n)$.

As for the usual permutations, one can generate a graph which vertices are the elements of $\mathfrak{P}(n)$ and which edges are defined according to the following operators:

- The operator M_i acts on the i-th and the $i+1$-th parentheses of a pseudo-permutation as follows: if each element of the i-th parenthese is smaller than all the elements of the $(i+1)$-th, then one can merge these two parentheses into one single parenthese which contains the union of the elements of these two parentheses.

- The operator $S_{i,j}$ acts on the i-th parenthese of a pseudo-permutation as follows: it splits this parenthese into two parentheses, the *second* one containing the j smallest elements of the initial parenthese and the first one containing the others.

For example, one can apply M_2, M_4, $S_{3,1}$, $S_{3,2}$ and $S_{4,1}$ to $\sigma = (7)(3)(568)(12)(4)$: $\sigma \xrightarrow{M_2} (7)(3568)(12)(4)$, $\sigma \xrightarrow{M_4} (7)(3)(568)(124)$, $\sigma \xrightarrow{S_{3,1}} (7)(3)(68)(5)(12)(4)$, $\sigma \xrightarrow{S_{3,2}} (7)(3)(8)(56)(12)(4)$ and $\sigma \xrightarrow{S_{4,1}} (7)(3)(568)(2)(1)(4)$.

We can now define the *pseudo-permutohedron*, denoted by $\mathfrak{G}(n)$, as the directed graph one obtains by iterating these two operators from the element $(1)(2)\cdots(n)$. For example, we give in Figure 1 the graphs $\mathfrak{G}(3)$ and $\mathfrak{G}(4)$.

We will now see some properties of the pseudo-permutations and their graph, first from the classical combinatorial point of view, then from the lattice point of view.

4 Combinatorial properties of $\mathfrak{P}(n)$

In this section, we give enumeration results, combinatorial properties and structural properties of the set $\mathfrak{P}(n)$ and its graph. Let us begin with the

[1] A *partition* of a set S is a set of subsets S_1, S_2, ..., S_k of S such that $\cup_{i=1}^{k} S_i = S$ and for all $i \neq j$, $S_i \cap S_j = \emptyset$. The S_i are then the *parts* of the partition. A partition is *ordered* if we consider a sequence of subsets instead of a set of subsets.

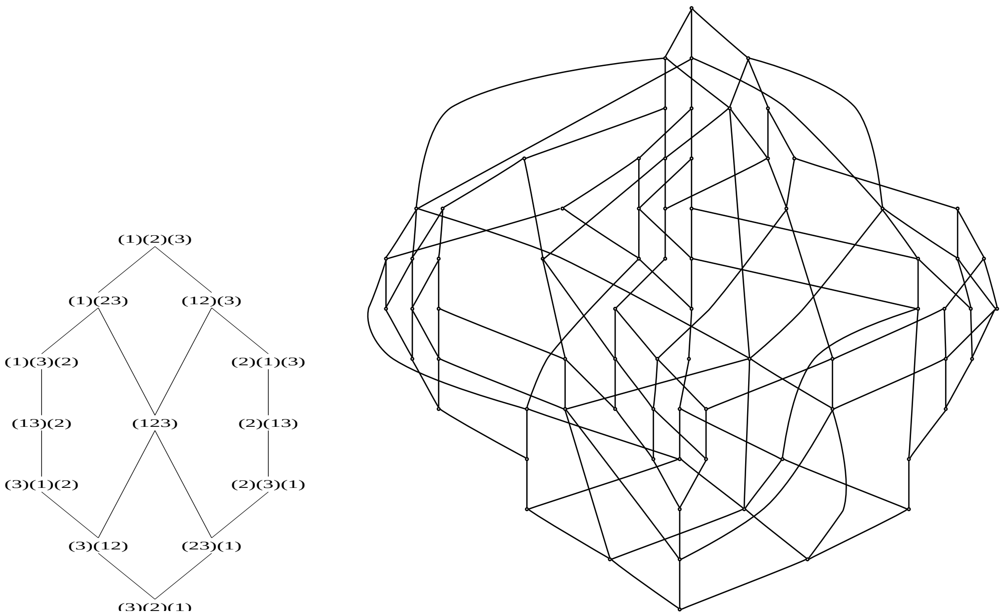

Figure 1: The graphs $\mathfrak{G}(3)$ (left) and $\mathfrak{G}(4)$ (right). The orientation of the edges is not displayed since it is always from the topmost vertex to the other.

most simple enumerations. We remind to the reader that we will not give the full proofs of the properties since they are long, technical, and do not give any further insight in the structure of pseudo-permutations. They will be published in the long version of the paper. However, we will always give a few hints about the proofs so that the reader can believe in the claims.

4.1 Enumeration of $\mathfrak{P}(n)$

As defined, the set $\mathfrak{P}(n)$ is the set of ordered partitions of n elements with nonempty parts. Therefore, it satisfies the induction relation:

$$\mathrm{Card}(\mathfrak{P}(n)) = \sum_{i=0}^{n-1} \binom{n}{i} \mathrm{Card}(\mathfrak{P}(i)).$$

One can also transform the set $\mathfrak{P}(n)$ in order to find a direct (well-known) formula, related to the Eulerian numbers.

Let n be an integer. As already noticed, the order of the elements in a given parenthese is irrelevant, so we can assume that they are written in increasing order. Given an element σ of $\mathfrak{P}(n)$, one can then define its *corresponding permutation* as the permutation obtained by removing all the parentheses. For example, the corresponding permutation of (1)(34)(2)(5) is (13425).

One can then consider the classes of the pseudo-permutations which have the same corresponding permutation: for example, the pseudo-permutations

(123), (1)(23), (12)(3), and (1)(2)(3) form a class of $\mathfrak{P}(n)$. It is clear that, given a permutation σ, the number of pseudo-permutations in the corresponding class is to the number of correct sets of parentheses one can generate. These sets correspond to compositions of n and one can easily see that a composition $I = (i_1, \ldots, i_p)$ is correct if and only if the descent set of σ is included in the set $\{i_1, i_1 + i_2, \ldots, i_1 + \cdots, i_p\}$. It then comes that the number of correct parentheses for a given permutation σ of order n is equal to 2^{n-k-1} where k is the number of descents of σ.

Finally, since the number of pseudo-permutations generated from a permutation only depends on the order of the descent set of this permutation, one makes the connection with the Eulerian numbers and one derives the following formula:

$$\mathrm{Card}(\mathfrak{P}(n)) = \sum_{k=0}^{n-1} A_{n,k} 2^{n-k-1} = \sum_{k=0}^{n-1} A_{n,k} 2^k.$$

4.2 Inversions

In this subsection, we generalize to the pseudo-permutations the well-known statistics of inversions over usual permutations.

4.2.1 Table of inversions

Let n be an integer and let τ be an element of $\mathfrak{P}(n)$. For every pair (i, j) with $1 \leq i, j \leq n$ and $i \neq j$, we define the value of the inversion (i, j) as follows:

- If i and j are in the same parenthese, this value is equal to $\frac{1}{2}$.
- If i and j are in distinct parentheses, this value is equal to 0 if the parenthese of the smallest integer is before the parenthese of the greatest one in τ and it is equal to 1 in the other case.

The *table of inversions* of τ is then the list of the non zero-valued pairs (i, j) with $1 \leq i < j \leq n$ of integers given with the value of the inversion (i, j). The *number of inversions* of τ is then the sum for all $1 \leq i < j \leq n$ of the inversions (i, j).

For example, the table of inversions of the pseudo-permutation (54)(31)(2) is

$$\left\{\frac{1}{2}(1,3),\ (1,4),\ (1,5),\ (2,3),\ (2,4),\ (2,5),\ (3,4),\ (3,5),\ \frac{1}{2}(4,5)\right\},$$

and its number of inversions is 8.

4.2.2 Characterization of the inversion sets

One can now ask about the reverse problem: given an inversion table, does it exist a pseudo-permutation that has this inversion table and, when the answer is yes, how can one rebuild the corresponding pseudo-permutation.

As for the usual permutations, the answer to the first question is negative in general and one can get the following characterization which generalizes the characterization of the correct sets for the usual inversion table:

Theorem 4.1 *Let n be an integer and L be a list of the pairs (i, j), with $1 \leq i < j \leq n$, each pair having a coefficient in the set $\{0, \frac{1}{2}, 1\}$. Then L is the inversion table of a pseudo-permutation if and only if each of the following implications holds:*

- *If the coefficient of (i, j) and (i, k) is $\frac{1}{2}$ then the coefficient of (j, k) is also $\frac{1}{2}$.*
- *If the coefficients of (i, j) and (j, k) both are 1, and if $i < j < k$ then the coefficient of (i, k) is also 1.*
- *If the coefficient of (i, k) is 1, then for every $j \in [\![i, k]\!]$, either the coefficient of (i, j) or the coefficient of (j, k) is also 1.*
- *If the coefficient of (i, k) is $\frac{1}{2}$, then for every $j \in [\![i, k]\!]$, either the coefficient of (i, j), either the coefficient of (j, k) is 1, or both coefficients of (i, j) and (j, k) are $\frac{1}{2}$.*

Proof — We will not give the complete proof of this theorem but let us see why all these conditions are necessary:

- The first condition only says that if i, j and i, k are in the same parenthese then so are j and k.
- The second condition only says that if j precedes i and k precedes j, then k precedes i.
- The third condition says that if k precedes i then, whatever the position of j is, either k precedes j or j precedes i.
- The fourth condition does the same as the third one when k is in the same parenthese as i.

The end of the proof rebuilds from a set that satisfies the four conditions the corresponding pseudo-permutation. This step is very technical, so it is omitted here. ∎

With this notion of table of inversions, one can derive a natural partial order on pseudo-permutations: σ is smaller that τ if the table of inversions of σ is included in the table of τ.

We then define the *graph of inversions* of all the elements of $\mathfrak{P}(n)$ as the covering relation of this poset, i.e. the transitive and reflexive reduction of the order relation. The next theorem makes the connection between this graph and the pseudo-permutohedron, which gives much information on $\mathfrak{S}(n)$.

Theorem 4.2 *The graph of inversions of $\mathfrak{P}(n)$ is the same as its pseudo-permutohedron.*

Proof — The proof of this theorem is very technical and we will omit it since we will only need its statement in the rest of the paper. ■

4.3 Descents of elements of $\mathfrak{P}(n)$

Let σ be a pseudo-permutation. We define its *descent set* and its *descent number* respectively as the descent set and the descent number of its corresponding permutation. We define in the same way the *rise number* of a pseudo-permutation. One then has the following properties:

Proposition 4.3 *Let σ be a pseudo-permutation. Then the number of incoming edges of σ in the pseudo-permutohedron is equal to the descent number of σ.*

Proposition 4.4 *Let σ be a pseudo-permutation. Then the number of outgoing edges of σ in the pseudo-permutohedron is equal to the rise number of σ.*

Proof — These two propositions can be proved in the same way, first showing that the parentheses play disjoint roles, so that one can concentrate on a given parenthese, then studying the case of a single pair of parentheses and end the proof by induction. ■

With the help of one of the previous properties, one can easily establish the following enumeration result:

Corollary 4.5 *Let n be an integer. The number of edges of the pseudo-permutohedron of order n is given by:*

$$\sum_{k=0}^{n-1} A_{n,k} 2^k k.$$

5 Lattice properties of $\mathfrak{P}(n)$

5.1 $\mathfrak{G}(n)$ is a lattice

In this subsection, we prove that the pseudo-permutohedron is a lattice. To prove this property, we first define the infimum and the supremum of two given elements. We actually use the fact that $\mathfrak{G}(n)$ is nothing but the inversion graph of $\mathfrak{P}(n)$ (Theorem 4.2), and we prove that this last graph is the covering relation of a lattice.

To acheive this, we introduce the trans-union of two given pseudo-psemutations. First consider the table of inversions of these elements. Then make the union of these tables, taking the maximum value for each pair (i, j). Then compute the transitive closure of this table, that is the smallest table that contains the union and satisfies the first two properties of Theorem 4.1. This last table is the *trans-union* of the two elements. For example, if we consider (1)(23) and (12)(3) with inversion tables respectivly $\{\frac{1}{2}(2,3)\}$ and $\{\frac{1}{2}(1,2)\}$, their trans-union is $\{\frac{1}{2}(1,2), \frac{1}{2}(2,3), \frac{1}{2}(1,3)\}$. Then one has:

Proposition 5.1 *Let us consider two elements of a pseudo-permutohedron. Then their trans-union table is the inversion table of a pseudo-permutation that belong to the same pseudo-permutohedron. Moreover, this element is their infimum.*

Proof — This proof has two parts that are not very complicated but quite long and techincal. First, one checks that the trans-union satisfies the four properties of Theorem 4.1, so that it is the inversion table of a pseudo-permutation. Then one proves that this table is included in the one of each element greater than the two elements, and so it is the greatest such element (by definition of the order over pseudo-permutations in the inversion graph), i.e. the infimum.■

As it is the case for other lattices, one cannot define the supremum of two elements in such a simple way. However, the pseudo-permutohedron has an interesting property we can use for this. If we say that the dual of an element is obtained by reading it from right to left, then the graph obtained from $\mathfrak{G}(n)$ by replacing each pseudo-permutation by its dual and by reversing each edge is $\mathfrak{G}(n)$ itself. $\mathfrak{G}(n)$ is then said to be *auto-dual*. Then, we obviously have:

Proposition 5.2 *Let us consider two elements of a pseudo-permutohedron. Their* supremum *is the dual of the infimum of their duals.*

We can therefore state the main result of this section:

Theorem 5.3 *For all integer n, the pseudo-permutohedron of order n is a lattice.*

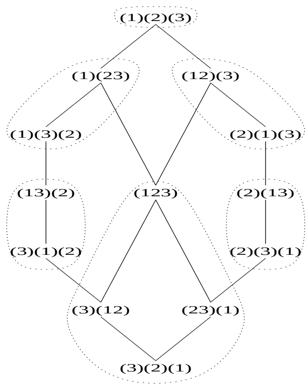

Figure 2: $\mathfrak{S}(3)$ is a disjoint union of hypercubes.

5.2 Cutting the pseudo-permutohedron

Let us consider a pseudo-permutohedron and let us consider the connected components obtained by removing the edges which correspond to the S operators. Then each connected component is a hypercube that contains all the elements of the pseudo-permutohedron which have the same corresponding permutation. The top-most element is built from the permutation by cutting it as the concatenation of the smallest possible number of decreasing sequences and putting these sequences into the same parentheses. For example, if the corresponding permutation is (7614352) then the top-most element is (761)(43)(52), i.e. (167)(34)(25).

The dimension of the hypercube is equal to the number of descents of its corresponding permutation, i.e. the number of ingoing edges of its elements. As an example, we give the decomposition of $\mathfrak{P}(3)$ as an union of hypercubes in Figure 2.

As we saw before, the pseudo-permutohedron is auto-dual, therefore all the properties one can establish for it can be dualized. Since the C edges and the E edges are exchanged in this process, one can derive that the number of C edges is the half of the total number of edges of $\mathfrak{S}(n)$. The previous results then allows to make in another way the enumeration of the edges of $\mathfrak{S}(n)$.

Conclusion

In this paper, we have shown a few properties of a new combinatorial object. These properties are very essential since they are generalizations of the key properties of the symmetric group, which allows us to think that many other statistics defined on the symmetric group have a real meaning in this context. This paper only opens the way to other combinatorial and lattice results about the symmetric group in our context to reach a real analog of the permutohedron studies. We believe that the pseudo-permutohedron is an interesting and meaningful extension of the permutohedron, and we hope that some work will be done to extend the classical results to this case, for example the geometrical interpretation of the permutohedron.

Another interesting direction of investigation with many applications is to consider pseudo-permutations with multiple occurences of the integers. This gives a very general and powerful model which makes it possible to represent the relations between multiple events which can have a temporal length. It seems that most of the results presented here are still valid in this context, and some new questions arise.

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