The lattice of integer partitions and its infinite extension

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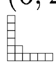
Abstract: In this paper, we use a simple discrete dynamical model to study integers partitions and their lattice. The set of reachable configurations of the model, with the order induced by the transition rule is exactly the lattice of all the partitions of an integer, equiped with the dominance ordering. We first explain how this lattice can be constructed, by showing its self-similarity property. Then, we define a natural extension of the model to infinity, which is compared to the Young lattice. Using a self-similar tree, we obtain an efficient encoding of the obtained lattice which makes it possible to compute easily and efficiently all the partitions of a given integer. This approach also gives a new formula for the number of partitions of an integer, and some informations on special sets of partitions, such as length bounded partitions.

Keywords: Lattice, Dominance ordering, Integer partitions, Sand Pile Model, Young lattice, Discrete Dynamical Models.

1. Preliminaries

A *partially ordered set* (or *poset*) is a set P with a reflexive ($x \leq x$), transitive ($x \leq y$ and $y \leq z$ implies $x \leq z$) and antisymetric ($x \leq y$ and $y \leq x$ implies $x = y$) binary relation $\leq$. A *lattice* is a partially ordered set such that any two elements a and b have a least upper bound, called *supremum* of a and b and denoted by $a \vee b$, and a greatest lower bound, called *infimum* of a and b and denoted by $a \wedge b$. The element $a \vee b$ is the smallest element among the elements greater than both a and b. The element $a \wedge b$ is defined dually. A lattice is a strongly structured set, and many general results, for example efficient encodings and algorithms, are known about them. For more details, see for example [DP90].

A *partition* of the integer n is a k-tuple $a = (a_1, a_2, \ldots, a_k)$ of positive integers such that $\sum_{i=1}^{k} a_i = n$ and $a_i \geq a_{i+1}$ for all i between 1 and k (with the assumption that $a_{k+1} = 0$). The Ferrer diagram of a partition $a = (a_1, a_2, \ldots, a_k)$ is a drawing of a on k adjacent columns such that the i-th column is a pile of a_i stacked squares, which we will call *grains* because of the sand pile dynamics we will consider over them. For example, $p = (4, 3, 3, 2)$ and $q = (6, 2, 1, 1, 1, 1)$ are two partitions of $n = 12$, and their Ferrer diagrams are  and  respectively.

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The *dominance ordering* is defined in the following way [Bry73]. Consider two partitions of the integer n: $a = (a_1, a_2, \dots, a_k)$ and $b = (b_1, b_2, \dots, b_l)$. Then

$$a \geq b \text{ if and only if } \sum_{i=1}^{j} a_i \geq \sum_{i=1}^{j} b_i \text{ for all } j.$$

From [Bry73], it is known that the set of all the partitions of an integer n with the dominance ordering is a lattice, denoted by $L_B(n)$. In his paper, Brylawski proposed a dynamical approach to study this lattice. We will introduce a few notations to explain it intuitively. For more details about integer partitions, we refer to [And76].

Let $a = (a_1, \dots a_k)$ be a partition. The *height difference of a at i*, denoted by $d_i(a)$, is the integer $a_i - a_{i+1}$ (with the assumption that $a_{k+1} = 0$). We say that the partition a has a *cliff* at i if $d_i(a) \geq 2$. We say that a has a *slippery plateau* at i if there exists $k > i$ such that $d_j(a) = 0$ for all $i \leq j < k$ and $d_k(a) = 1$. The integer $k - i$ is then called the *length* of the slippery plateau at i. Likewise, a has a *non-slippery plateau* at i if $d_j(a) = 0$ for all $i \leq j < k$ and it has a cliff at k. The integer $k - i$ is called the *length* of the non-slippery plateau at i. The partition a has a *slippery step* at i if the partition defined by $a' = (a_1, \dots, a_i - 1, \dots, a_k)$ (if it exists) has a slippery plateau at i. Likewise, a has a *non-slippery step* at i if a' has a non-slippery plateau at i. See Figure 1 for some illustrations.

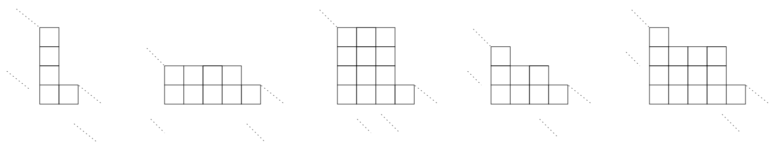

FIGURE 1. From left to right: a cliff, a slippery plateau of length 3, a non-slippery plateau of length 2, a slippery step of length 2 and a non-slippery step of length 3.

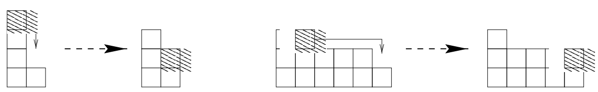

FIGURE 2. The two evolution rules of the dynamical model

Consider now the partition $a = (a_1, a_2, \dots, a_k)$. Brylawski defined the two following evolution rules: one grain can fall from column i to column $i + 1$ if a has a cliff at i, and one grain can slip from column i to column $i + l + 1$ if a has a slippery step of length l at i. See Figure 2.

Such a fall or a slip is called a *transition* of the model and is denoted by $a \xrightarrow{i} b$ where i is the column from which the grain falls or slips. If one starts from the partition (n) and iterates this operation, one obtains all the partitions of n, and the dominance ordering is nothing but the reflexive and transitive closure of the relation induced by the transition rule [Bry73]. See Figure 3 for examples with $n = 7$ and $n = 8$. We denote by $dirreach(a)$ the set of configurations directly reachable from a, *i.e.* the set $\{b \mid a \xrightarrow{i} b\}$. Notice that in the context of dynamical models theory, those elements are called *the immediate successors* of a. However, since we are concerned here with orders theory, we cannot use this term, which takes another meaning in this context.

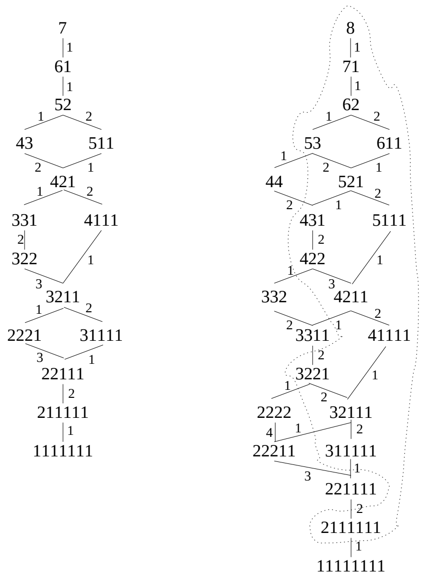

FIGURE 3. Diagrams of the lattices $L_B(n)$ for $n = 7$ and $n = 8$. As we will see, the set $L_B(7)$ is isomorphic to a sublattice of $L_B(8)$. On the diagram of $L_B(8)$, we included in a dotted line this sublattice.

Before entering in the core of the topic, we need one more notation. If the k-tuple $a = (a_1, a_2, \ldots, a_k)$ is a partition, then the k-tuple $(a_1, a_2, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots, a_k)$ is denoted by $a^{\downarrow i}$. In other words, $a^{\downarrow i}$ is obtained from a by adding one grain on its i-th column. Notice that the k-tuple obtained this way is not necessarily a partition. Finally, if S is a set of partitions, then $S^{\downarrow i}$ denotes the set $\{a^{\downarrow i} | a \in S\}$.

We will now study the structure of the lattice of the partitions of an integer n and we will show its self-similarity by giving a method to construct $L_B(n+1)$ from $L_B(n)$. Then, we will define an infinite extension of these lattices: the lattice $L_B(\infty)$ of all the partitions of any integer. This lattice has some interesting properties, which we will discuss. We will also compare it to the Young lattice, which also contains all the partitions of any integer, but ordered in a different way. Finally, we will construct a tree based on the construction process detailed in the beginning of the paper. This tree will make it possible to give a simple and efficient algorithm to compute all the partitions of a given integer. It also has a recursive structure, from which we will obtain new formula for the number of partitions of an integer n and some results about certain classes of partitions.

2. From $L_B(n)$ to $L_B(n+1)$

In this section, our aim is to construct $L_B(n+1)$ from $L_B(n)$, viewed as the graph induced by the dynamical model, with the edges labelled by the number of the column from which the grain falls or slips, as shown in Figure 3. We will call *construction of a lattice* the computation of this labelled graph. We first show that $L_B(n)^{\downarrow 1}$ is a sublattice of $L_B(n+1)$. For example, in Figure 3 we included in a dotted line $L_B(7)^{\downarrow 1}$ within $L_B(8)$. This remark allows us to start the construction of $L_B(n+1)$ from $L_B(n)$ by computing $L_B(n)^{\downarrow 1}$ and then adding the missing elements of $L_B(n+1)$. After characterizing those elements that must be added, we obtain a simple and efficient method to acheive the construction of $L_B(n+1)$ from $L_B(n)$.

Proposition 1. *$L_B(n)^{\downarrow 1}$ is a sublattice of $L_B(n+1)$.*

Proof. We must show that $\inf(a,b) = c$ in $L_B(n)$ implies $\inf(a^{\downarrow 1}, b^{\downarrow 1}) = c^{\downarrow 1}$ in $L_B(n+1)$, and that $\sup(a,b) = c$ in $L_B(n)$ implies $\sup(a^{\downarrow 1}, b^{\downarrow 1}) = c^{\downarrow 1}$ in $L_B(n+1)$. We know from [Bry73] that the dominance ordering over $L_B(n)$ implies:

$$\inf(a,b) = c \text{ if and only if, for all } j, \text{ one has } \sum_{i=1}^{j} c_i = min(\sum_{i=1}^{j} a_i, \sum_{i=1}^{j} b_i).$$

From this, it is straightforward to see that $c^{\downarrow 1}$ is in $L_B(n+1)$, and clearly $c^{\downarrow 1} = \inf(a^{\downarrow 1}, b^{\downarrow 1})$.

Let now $c = \sup(a,b)$ in $L_B(n)$ and $d = \sup(a^{\downarrow 1}, b^{\downarrow 1})$ in $L_B(n)^{\downarrow 1}$. We will show that $d = c^{\downarrow 1}$. We have $c \geq a$ and $c \geq b$, therefore $c^{\downarrow 1} \geq a^{\downarrow 1}$ and $c^{\downarrow 1} \geq b^{\downarrow 1}$. This implies that $c^{\downarrow 1} \geq d$. To show that $d \geq c^{\downarrow 1}$, let us begin by showing that $d_1 = c_1 + 1$. We can suppose $a_1 \geq b_1$. The partition $(a_1, a_1, a_1 - 1, a_1 - 2, \dots)$ is greater than a and b, and so it is greater than c. Moreover, $c \geq a$ implies $c_1 \geq a_1$ and so $c_1 = a_1$. Since $a^{\downarrow 1} \leq d \leq c^{\downarrow 1}$, we then have $d_1 = a_1 + 1 = c_1 + 1$. Let $e = (d_1 - 1, d_2, d_3, \dots)$. Since $d \leq c^{\downarrow 1}$ and $c_1 = a_1$, e is a partition: $d_1 - 1 \geq d_2$. Moreover, $d \geq a^{\downarrow 1}$ and $d \geq b^{\downarrow 1}$, and so $e \geq a$ and $e \geq b$. This implies that $e \geq \sup(a,b) = c$ and that $d \geq c^{\downarrow 1}$, which ends the proof. □

This result shows that one can construct the lattice $L_B(n+1)$ from $L_B(n)$as follows. The first step of this construction is to construct the set $L_B(n)^{\downarrow 1}$ by adding one grain to the first column of each element of $L_B(n)$. Then, one has to add the missing elements and their edges. Therefore, we will now consider the consequences of the addition of one grain on the first column of a partition, depending on its structure.

Proposition 2. *Let a be a partition. Then, we have:*

1. *if a has a cliff or a non-slippery plateau at* 1 *then:*
$$dirreach(a^{\downarrow 1}) = dirreach(a)^{\downarrow 1}$$
2. *if a has a slippery plateau of length l at* 1 *then* $a^{\downarrow 1} \xrightarrow{1} a^{\downarrow l+1}$ *and:*
$$dirreach(a^{\downarrow 1}) = dirreach(a)^{\downarrow 1} \cup \{a^{\downarrow l+1}\}$$

3. *if a has a slippery step at 1, then let b be such that $a \xrightarrow{1} b$. We have* $a^{\downarrow_1} \xrightarrow{1} a^{\downarrow_2} \xrightarrow{2} b^{\downarrow_1}$ *and:*
$$dirreach(a^{\downarrow_1}) = (dirreach(a) \setminus \{b\})^{\downarrow_1} \cup \{a^{\downarrow_2}\}$$

Proof. 1. Notice first that the transitions possible from a on columns other than the first one are still possible from $a^{\downarrow_1}$. It suffices now to see that the addition of one grain on a cliff does not allow any new transition from the first column, since such a transition was already possible. Likewise, the addition of one grain on a non-slippery plateau does not allow a new transition.
2. No transition from a was possible on its first column. However, $a^{\downarrow_1}$ has a slippery step at 1 and so a transition $a^{\downarrow_1} \xrightarrow{1} b$ is now possible. It is easy to verify that $b = a^{\downarrow_{l+1}}$.
3. Notice first that transitions on columns other than the first one are not affected by addition of one grain on the first column. Let us now observe what happens about the transition from the first column. Notice that one transition from the first column is still possible, since $a^{\downarrow_1}$ has a cliff at column 1. However, the transition $a^{\downarrow_1} \xrightarrow{1} b^{\downarrow_1}$ is now impossible, but a new transition is possible: $a^{\downarrow_1} \xrightarrow{1} a^{\downarrow_2}$. Now, $a^{\downarrow_2}$ has either a slippery step or a cliff at 2, and so $b^{\downarrow_1}$ is directly reachable from it. □

For a given integer n, let us denote by $S(n)$ the set of partitions of n with a slippery step at 1, by $T(n)$ the set of partitions of n with a non-slippery step at 1, and by $U_l(n)$ the set of partitions of n with a slippery plateau of length l at 1. The propositions above show that, once we have $L_B(n)^{\downarrow_1}$, the next step of the construction is the addition of the elements and edges of the sets $S(n)^{\downarrow_2}$, $T(n)^{\downarrow_2}$ and $U_l(n)^{\downarrow_{l+1}}$. Now, we must add the missing transitions from these new elements, and the missing elements directly reachable from them. We show below that actually no element is missing, and we give a description of which missing transitions need to be added.

Theorem 1. *Every element of $L_B(n+1)$ is in $L_B(n)^{\downarrow_1}$, in $S(n)^{\downarrow_2}$, in $T(n)^{\downarrow_2}$ or in $U_l(n)^{\downarrow_{l+1}}$ for some l.*

Proof. We have shown that all the configurations directly reachable from the elements of $L_B(n)^{\downarrow_1}$ are in the union of these sets. Let us now show that all the configurations directly reachable from the elements of the union are already in it. Several cases are possible.

• Let $a \in S(n)$. Then, $a \xrightarrow{1} b$ in $L_B(n)$. Only one of the possible transitions from a is affected by the addition of one grain on the second column: the transition on column 1. Moreover, due to the choice of a in $S(n)$, a transition $\xrightarrow{2}$ is possible from $a^{\downarrow_2}$. From these remarks, we obtain:

$$dirreach(a^{\downarrow_2}) = (dirreach(a) \setminus \{b\})^{\downarrow_2} \ \cup \ \{b^{\downarrow_1}\}.$$

Notice now that the elements of $dirreach(a)\setminus\{b\}$ have a slippery step at 1, therefore the elements from $(dirreach(a) \setminus \{b\})^{\downarrow_2}$ are in $S(n)^{\downarrow_2}$. Moreover, $b^{\downarrow_1}$ is in $L_B(n)^{\downarrow_1}$, therefore there is no missing element directly reachable from $a^{\downarrow_2}$, and the transition $a^{\downarrow_1} \xrightarrow{1} b^{\downarrow_1}$ is replaced by $a^{\downarrow_2} \xrightarrow{2} b^{\downarrow_1}$.

• Let $a \in T(n)$. Then, the addition of one grain on the second column of a does not prevent any transition, and we have:

$$dirreach(a^{\downarrow_2}) = dirreach(a)^{\downarrow_2}.$$

Moreover, the elements of $dirreach(a)$ have a slippery or non-slippery step at 1, and therefore the elements of $dirreach(a)^{\downarrow_2}$ and the transitions from $a^{\downarrow_2}$ are in $S(n)^{\downarrow_2} \cup T(n)^{\downarrow_2}$.

• Let $a \in U_l(n)$. This case requires more attention. We distinguish three subcases:

1. a has a cliff at $l+1$. Then, we have $a \xrightarrow{l+1} b \xrightarrow{l} c$ in $L_B(n)$. The addition of one grain on the $(l+1)$-th column of a does not prevent any transition, therefore we have:

$$dirreach(a^{\downarrow_{l+1}}) = dirreach(a)^{\downarrow_{l+1}}.$$

   From the choice of a, we know that the elements of $dirreach(a) \setminus \{b\}$ have a slippery plateau at 1. Therefore all the elements of $dirreach(a)^{\downarrow_{l+1}}$ have already been added. Moreover, one can verify that $c = (a_1, \ldots, a_l - 1, a_{l+1}, a_{l+2} + 1, \ldots)$, and that $a^{\downarrow_1} \xrightarrow{1} a^{\downarrow_{l+1}} \xrightarrow{l+1} c^{\downarrow_l}$. Therefore, $b^{\downarrow_{l+1}} = c^{\downarrow_l}$. Moreover, c has a slippery plateau of length $l-1$ at 1, therefore the element $c^{\downarrow_l}$ has already been added. Thus, no element is missing; there is only one missing transition: $a^{\downarrow_{l+1}} \xrightarrow{l+1} b^{\downarrow_{l+1}}$.
2. a has a non-slippery step at l, and so a non-slippery plateau of length l' at $l+1$ (with possibly $l' = 0$). Then, the addition of one grain on the $(l+1)$-th column of a does not prevent any transition that was previously possible. Therefore, we have:

$$dirreach(a^{\downarrow_{l+1}}) = dirreach(a)^{\downarrow_{l+1}}.$$

   The elements of $dirreach(a)$ all have a non-slippery plateau at 1, therefore all the elements of $dirreach(a)^{\downarrow_{l+1}}$ have already been added.
3. a has a slippery step of length l' at l, and so a slippery plateau of length l' at $l+1$ (with possibly $l' = 0$). Then, $a \xrightarrow{l} b$ in $L_B(n)$. The possible transitions from $a^{\downarrow_{l+1}}$ are the same as the possible ones from a, except the transition on the column l. All the elements directly reachable from a except b have a slippery plateau at 1, therefore the elements of $dirreach(a) \setminus \{t\}^{\downarrow_{l+1}}$ have already been added. Moreover, $a^{\downarrow_{l+1}} \xrightarrow{l+1} a^{\downarrow_{l+l'+1}}$. But we can verify that $a^{\downarrow_{l+l'+1}} = b^{\downarrow_l}$, and, since b has a slippery plateau of length $l-1$ at 1, this element has already been added; there is only one missing transition: $a^{\downarrow_{l+1}} \xrightarrow{l+1} b^{\downarrow_l}$.

□

This result makes it possible to write an algorithm which constructs the lattice $L_B(n+1)$ from $L_B(n)$ in linear time with respect to the number of added elements and transitions. Notice that we can obtain $L_B(n)$ for an arbitrary integer n by starting from $L_B(0)$ and iterating this algorithm, and so we have an algorithm that constructs $L_B(n)$ in linear time with respect to its size.

3. The infinite lattice $L_B(\infty)$

We will now define $L_B(\infty)$ as the set of all the configurations reachable from (∞) (this is the configuration where the first column contains infinitely many grains and all the other columns contain no grain). Therefore, each element a of $L_B(\infty)$ has the form $(\infty, a_2, a_3, \ldots, a_k)$. As in the previous section, the dominance ordering on $L_B(\infty)$ (when the first component is ignored) is equivalent to the order induced by the dynamical model. The first partitions in $L_B(\infty)$ are given in Figure 4 along with their covering relations (the first component, equal to ∞, is not represented on this diagram).

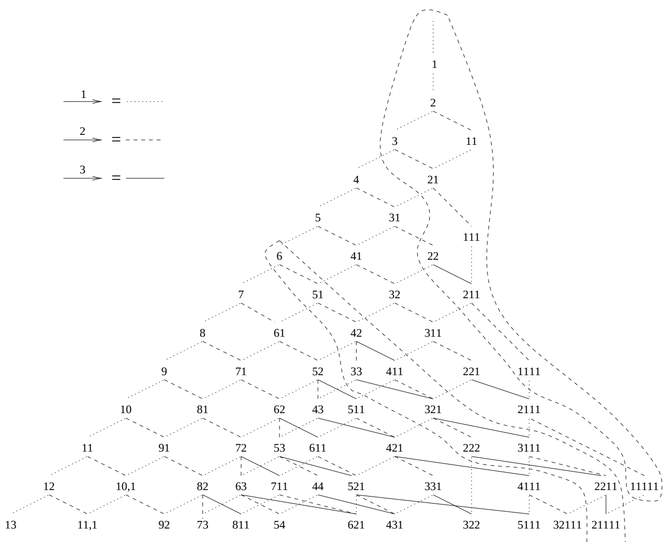

FIGURE 4. The first elements and transitions of $L_B(\infty)$. As shown on this figure for $n = 6$, we will see two ways to find parts of $L_B(\infty)$ isomorphic to $L_B(n)$ for any n.

It is easy to see that we have a characterization of the order similar to the one given in [Bry73] for the finite case: let a and b be two elements of $L_B(\infty)$, a being of length p and b being of length q. Then,

$$a \geq_{L_B(\infty)} b \text{ if and only if for all } j \text{ between 2 and } \max(p,q), \sum_{i \geq j} a_i \leq \sum_{i \geq j} b_i.$$

We will start this section by proving that $L_B(\infty)$ is a lattice and by giving a formula for the infimum in $L_B(\infty)$. After this, we will show that, for any n, there are two different ways to find sublattices of $L_B(\infty)$ isomorphic to $L_B(n)$. We will also give a way to compute some other special sublattices of $L_B(\infty)$, using its self-similarity. Finally, we will compare $L_B(\infty)$ to the Young lattice.

Theorem 2. *The set $L_B(\infty)$ is a lattice. Moreover, if $a = (\infty, a_2, \ldots, a_k)$ and $b = (\infty, b_2, \ldots, b_l)$ are two elements of $L_B(\infty)$, then $\inf_{L_B(\infty)}(a, b) = c$ in $L_B(\infty)$, where c is defined by:*

$$c_i = max(\sum_{j\geq i} a_j, \sum_{j\geq i} b_j) - \sum_{j>i} c_j \quad \text{for all } i \text{ such that } 2 \leq i \leq max(k, l).$$

Proof. We first prove that c is an element of $L_B(\infty)$ and then we prove that c is equal to $inf_{L_B(\infty)}(a, b)$. Let $n = 2(\sum_{i\geq 2} a_i + \sum_{i\geq 2} b_i)$. Let $a' = (n - \sum_{i\geq 2} a_i, a_2, \ldots, a_k)$, $b' = (n - \sum_{i\geq 2} b_i, b_2, \ldots, b_l)$ and $c' = (n - \sum_{i\geq 2} c_i, c_2, \ldots, c_{max(k,l)})$. It is then obvious that a' and b' are two partitions of n and that c' is the infimum of a' and b' by dominance ordering in $L_B(n)$. Therefore, c' is a decreasing sequence, and so c is an element of $L_B(\infty)$. Moreover, according to the definition of $\geq_{L_B(\infty)}$, c is the maximal element of $L_B(\infty)$ which is smaller than a and b, and so $c = inf_{L_B(\infty)}(a, b)$.

By definition, $L_B(\infty)$ has a maximal element. Since it is closed for the infimum, $L_B(\infty)$ is a lattice. □

Theorem 3. *Let n be a positive integer. The application:*

$$\begin{array}{cccc} \pi: & L_B(n) & \longrightarrow & L_B(\infty) \\ & a = (a_1, a_2, \ldots, a_k) & \longrightarrow & \bar{a} = (\infty, a_2, \ldots, a_k) \end{array}$$

is a lattice embedding.

Proof. It is obvious that π is injective. Moreover, we can apply a proof similar to the one of Proposition 1 to show that $inf_{L_B(\infty)}(\pi(a), \pi(b)) = \pi(inf_{L_B(n)}(a, b))$ and $sup_{L_B(\infty)}(\pi(a), \pi(b)) = \pi(sup_{L_B(n)}(a, b))$. We can then conclude that π is a lattice embedding. □

Let $\overline{L_B(n)}$ denote the image by π of $L_B(n)$ in $L_B(\infty)$. From Theorem 3, $\overline{L_B(n)}$ is a sublattice of $L_B(\infty)$. From Proposition 1, $L_B(n)^{\downarrow 1}$ is a sublattice of $L_B(n+1)$. Therefore, since $\overline{L_B(n)^{\downarrow 1}} = \overline{L_B(n)}$, we have an increasing sequence of sublattices:

$$\overline{L_B(0)} \leq \overline{L_B(1)} \leq \cdots \leq \overline{L_B(n)} \leq \overline{L_B(n+1)} \leq \cdots \leq L_B(\infty).$$

where $\leq$ denotes the sublattice relation.

We can say more about this increasing sequence of lattices. Let $a = (\infty, a_2, a_3, \ldots, a_k)$ be an element of $L_B(\infty)$. If one takes $a_1 = a_2 + 1$ and $n = \sum_{i=1}^{k} a_i$, then the partition $a' = (a_1, a_2, \ldots, a_k)$ is an element of $L_B(n)$. Since $a = \pi(a')$, this implies that a is an element of $\overline{L_B(n)}$. Conversly, any element of $L_B(\infty)$ is of the form $a = (\infty, a_2, \ldots, a_k)$. Therefore, $a' = (a_2, \ldots, a_k)$ is a decreasing sequence, and if we put $n = \sum_{i\geq 2} a_i$ then $a' \in L_B(n)$, *i.e.* $a \in \overline{L_B(n)}$. Finally, we have:

$$\bigcup_{n\geq 0} \overline{L_B(n)} = L_B(\infty)$$

Therefore, $L_B(\infty)$ can be viewed as the limit of $L_B(n)$ when n grows to infinity.

Theorem 3 gives a way to find, for any n, a sublattice of $L_B(\infty)$ isomorphic to $L_B(n)$. We will see in the following another way to find such parts. In order to acheive this goal, we first study the infinite union of all the sets $L_B(n)$ for any n:

$$\widetilde{L_B(\infty)} = \bigsqcup_{n \geq 0} L_B(n)$$

where $\sqcup$ denotes the disjoint union. We consider the following relation over $\widetilde{L_B(\infty)}$. Let $a \in L_B(m)$ and $b \in L_B(n)$. We have $a \xrightarrow{i} b$ in $\widetilde{L_B(\infty)}$if and only if one of the following applies: $n = m$ and $a \xrightarrow{i} b$ in $L_B(n)$, or $i = 0$, $n = m + 1$ and $b = a^{\downarrow 1}$. In other terms, the elements of $L_B(n)$ are linked to each other as usual, whereas each element a of $L_B(n)$ is linked to $a^{\downarrow 1} \in L_B(n+1)$ by an edge labelled by 0.

From this, one can introduce an order on the set $\widetilde{L_B(\infty)}$ in the usual sense, by defining it as the reflexive and transitive closure of this relation. We now show that $L_B(\infty)$ is isomorphic to $\widetilde{L_B(\infty)}$, and so that $\widetilde{L_B(\infty)}$ is a lattice.

Theorem 4. *The application χ defined by:*

$$\chi : \widetilde{L_B(\infty)} \longrightarrow L_B(\infty)$$
$$a = (a_1, a_2, \dots, a_k) \mapsto \chi(a) = (\infty, a_1, a_2, \dots, a_k)$$

is a lattice isomorphism.
Moreover, $a \xrightarrow{i} b$ in $\widetilde{L_B(\infty)}$ if and only if $\chi(a) \xrightarrow{i+1} \chi(b)$ in $L_B(\infty)$.

Proof. χ is clearly bijective. Moreover, it is clear from the definitions that for all a and b in $\widetilde{L_B(\infty)}$, $a \xrightarrow{i} b$ if and only if $\chi(a) \xrightarrow{i+1} \chi(b)$. Therefore, χ is an order isomorphism. Since $L_B(\infty)$ is a lattice, this implies that χ is a lattice isomorphism.
□

This theorem means that $\widetilde{L_B(\infty)}$ is nothing but $L_B(\infty)$ when one removes the first component (always equal to ∞) of each element of $L_B(\infty)$ and decreases the label of each edge by 1. We will now see that $L_B(n)$ is a sublattice of $\widetilde{L_B(\infty)}$ for all n, which gives another way to find a part of $L_B(\infty)$ isomorphic to $L_B(n)$.

Theorem 5. *For all integer n, $L_B(n)$ is a sublattice of $\widetilde{L_B(\infty)}$.*

Proof. Let a and b be two elements of $L_B(n)$, we prove that $inf_{\widetilde{L_B(\infty)}}(a, b)$ and $sup_{\widetilde{L_B(\infty)}}(a, b)$ belong to $L_B(n)$. Let c be $inf_{L_B(n)}(a, b)$ and c' be $inf_{\widetilde{L_B(\infty)}}(a, b)$. We have, $a \geq_L c' \geq_{\widetilde{L_B(\infty)}} c$, which means that $\sum_{i \geq 1} a_i \leq \sum_{i \geq 1} c'_i \leq \sum_{i \geq 1} c_i$, and so $\sum_{i \geq 1} c'_i = n$. This implies that c' belongs to $L_B(n)$, and we obtain $c' = c$. The proof for the supremum is similar. □

We now have two different ways to find, for any integer n, a part of $L_B(\infty)$ isomorphic to $L_B(n)$. We can use this to compute some parts of $L_B(\infty)$. However, these methods do not give filters[2] of $L_B(\infty)$, which is however possible. We explain how in the following.

Notice first that $\widetilde{L_B(\infty)}$ can be viewed as the limit of the sequence of posets defined for any n by:

$$L_B(\leq n) \;=\; \bigsqcup_{0\leq i\leq n} L_B(i)$$

with the same relation as the one defined above for $\widetilde{L_B(\infty)}$. From Theorem 3 and 5, we can deduce an efficient method to construct $L_B(\leq n)$ for all n: it suffices to compute (recursively) $L_B(\leq n-1)$, extract from it the part $\overline{L_B(n)}$, deduce $L_B(n)$ from this, and then add the links of the set $\{a \xrightarrow{0} a^{\downarrow 1}$ such that $a \in \overline{L_B(n-1)}\}$. We obtain this way $L_B(\leq n)$. We show now that $L_B(\leq n)$ is a sublattice of $\widetilde{L_B(\infty)}$ for all n, which implies directly that it is a filter of $\widetilde{L_B(\infty)}$.

Proposition 3. *The poset $L_B(\leq n)$ is a sublattice of $\widetilde{L_B(\infty)}$ for all n.*

Proof. To prove that $L_B(\leq n)$ is a sublattice of $\widetilde{L_B(\infty)}$, we consider two elements a and b of $L_B(\leq n)$ and show that $inf_{\widetilde{L_B(\infty)}}(a,b)$ and $sup_{\widetilde{L_B(\infty)}}(a,b)$ belong to $L_B(\leq n)$. There exist k and l such that $a \in L_B(k)$ and $b \in L_B(l)$. We can suppose whithout loss of generality that $k \leq l \leq n$. Let $c = sup_{\widetilde{L_B(\infty)}}(a,b)$. Since $c \geq_{\widetilde{L_B(\infty)}} a$, there exists an integer $m \leq k$ such that $c \in L_B(m)$, and so $c \in L_B(\leq n)$. Let now $d = inf_{\widetilde{L_B(\infty)}}(a,b)$. Let $a' = (a_1 + l - k, a_2, \ldots)$. Then, a' is in $L_B(l)$ and $inf_{\widetilde{L_B(\infty)}}(a',b) \in L_B(l)$. Since $d = inf_{\widetilde{L_B(\infty)}}(a,b) \geq inf_{\widetilde{L_B(\infty)}}(a',b)$, we have $d \in L_B(\leq l)$. This implies the result. □

To finish this section, we will discuss the relations between the infinite lattice $\widetilde{L_B(\infty)}$ and the famous Young lattice. These two infinite lattices contain exactly the same elements (all the partitions of all the integers), but ordered in a different way: $a \leq b$ in the Young lattice if for all i we have $a_i \leq b_i$. In other words, the order over the partitions is the componentwise order. This order induces a (distributive) lattice structure over the set of all the integer partitions. It has been widely studied; see for example [Sta99, Ges93]. It can also be viewed as the set of partitions obtained from the empty one, (), and by iterating the following evolution rule: $a \xrightarrow{i} b$ if b is a partition obtained from the partition a by increasing its i-th component. This implies directly that the lattices can be decomposed into levels (the i-th level contains the partitions obtained after i applications of the evolution rule), and that level i contains exactly the partitions of n, *i.e.* the elements of $L_B(n)$. Notice moreover that these elements are not comparable in the Young lattice therefore the order in $\widetilde{L_B(\infty)}$ and the one in the Young lattice are very different. However, they are put in relation by the following theorem:

[2] A filter F of a poset P is a subset of P such that $\forall x \in F,\ \forall y \in P,\ y \geq x \Rightarrow y \in F$.

Theorem 6. [Lat00] *The application* π *from* $\widetilde{L_B(\infty)}$ *into the Young lattice such that* $\pi(a)_i$ *is equal to* $\sum_{j \geq i} a_j$ *is an order embedding which preserves the infimum.*

Proof. Let a and b be two elements of $\widetilde{L_B(\infty)}$. We must show that $\pi(a)$ and $\pi(b)$ belong to the Young lattice, that $a \geq b$ in $\widetilde{L_B(\infty)}$ is equivalent to $\pi(a) \geq \pi(b)$ in the Young lattice and that $\inf(\pi(a), \pi(b))$ in the Young lattice is equal to $\pi(\inf(a, b))$ in $\widetilde{L_B(\infty)}$. The two first points are easy: $\pi(x)$ is obviously a decreasing sequence of integers for any x, and the order is preserved. Now, let $c = \inf(a, b)$. Then,

$$\begin{aligned} \pi(c)_i &= \textstyle\sum_{j \geq i} c_j \\ &= max(\textstyle\sum_{j \geq i} a_j, \sum_{j \geq i} b_j) && \text{from Theorem 2} \\ &= max(\pi(a)_i, \pi(b)_i) \\ &= \inf(\pi(a), \pi(b))_i && \text{in the Young lattice} \end{aligned}$$

which proves the claim. □

Notice that this order embedding is not a *lattice* embedding, since it does not preserve the supremum. For example, if $a = (2, 2)$ and $b = (1, 1, 1)$, then $\pi(a) = (4, 2)$, $\pi(b) = (3, 2, 1)$, and $c = \sup(a, b) = (2, 1)$ in $\widetilde{L_B(\infty)}$ but $\pi(c) = (3, 1)$ and $\sup((4, 2), (3, 2, 1)) = (3, 2)$ in the Young lattice. Notice that there can be no lattice embedding from $\widetilde{L_B(\infty)}$ to the Young lattice since the fact this one is a *distributive* lattice would imply that $\widetilde{L_B(\infty)}$ would be distributive, which is not true. Finally, notice that a study similar to the one presented in this paper can be found in [Lat01] on another kind of integer partitions, namely b-ary partitions. The Young lattice is a particular case of the lattices introduced in this paper, and the reader interested in the relations between $\widetilde{L_B(\infty)}$ and the Young lattice should refer to it.

4. The infinite binary tree $T_B(\infty)$

As shown in our procedure to construct $L_B(n+1)$ from $L_B(n)$, each element a of $L_B(n+1)$ is obtained from an element a' of $L_B(n)$ by addition of one grain: $a = a'^{\downarrow i}$ for some integer i. We will now represent this relation by a tree where $a \in L_B(n+1)$ is the son of $a' \in L_B(n)$ if and only if $a = a'^{\downarrow i}$ and we label with i the edge $a' \longrightarrow a$ in this tree. We denote this tree by $T_B(\infty)$. The root of this tree is the empty partition (). We will show two ways to find the partitions of a given integer n in $T_B(\infty)$, which will make it possible to give an efficient and simple algorithm to compute them. Moreover, the recursive structure of this tree will allow us to obtain a recursive formula for the cardinal of $L_B(n)$ and some special classes of partitions.

From the construction of $L_B(n+1)$ from $L_B(n)$, it follows that the nodes of this tree are the elements of $\bigsqcup_{n \geq 0} L_B(n)$, and that each node a has at least one son, $a^{\downarrow 1}$, and one more if a begins with a slippery plateau of length l: the element $a^{\downarrow l+1}$. Therefore, $T_B(\infty)$ is a binary tree. We will call *left son* the first of two sons, and *right son* the other (if it exists). We call *the level* n *of the tree* the set of elements of depth n. The first levels of $T_B(\infty)$ are shown in Figure 5.

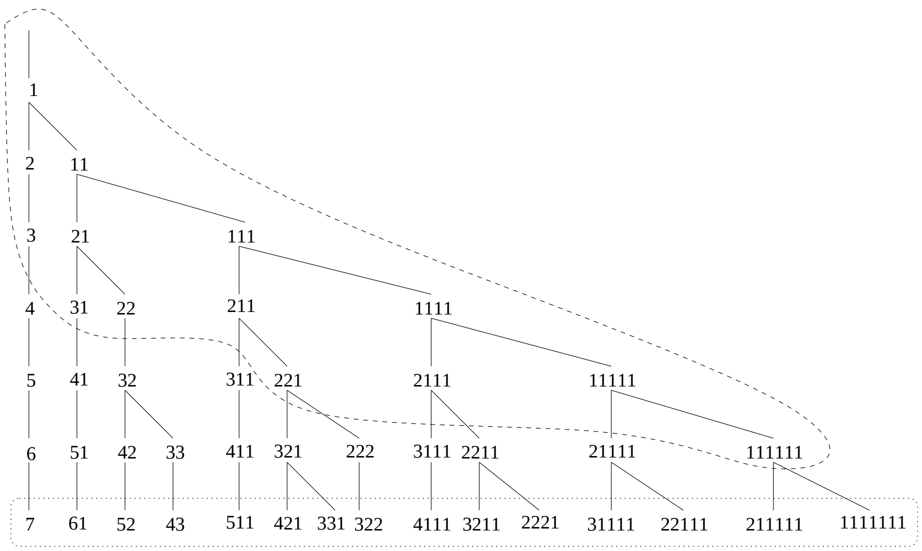

FIGURE 5. The first levels of the tree $T_B(\infty)$ (to clarify the picture, the labels are omitted). As shown on this figure for $n = 7$, we will see two ways to find the elements of $L_B(n)$ in $T_B(\infty)$ for any n.

Like in the case of $L_B(\infty)$, there are two ways to find the elements of $L_B(n)$ in $T_B(\infty)$. From the construction of $L_B(n+1)$ from $L_B(n)$ given above, it is straightforward that:

Proposition 4. *The level* n *of* $T_B(\infty)$ *is exactly the set of the elemens of* $L_B(n)$.

Moreover, it is obvious from the construction of $T_B(\infty)$ that the elements of the set $\overline{L_B(n+1)} \setminus \overline{L_B(n)}$ are sons of elements of $\overline{L_B(n)}$, therefore we deduce the following proposition which can easily be proved by induction:

Proposition 5. *Let* χ^{-1} *be the inverse of the lattice isomorphism defined in Theorem 4. Then, the set* $\chi^{-1}(\overline{L_B(n)})$ *is a subtree of* $T_B(\infty)$ *having the same root.*

This proposition makes it possible to give a simple and efficient algorithm to compute all the partitions of a given integer n in linear time with respect to their number. Indeed, it gives a binary tree structure to the set of all these partitions. See Algorithm 1.

We will now give a recursive description of $T_B(\infty)$. This will allow us to obtain a new recursive formula for $|L_B(n)|$, as well as for some special classes of partitions. We first define a certain kind of subtrees of $T_B(\infty)$. Afterwards, we show how the whole structure of $T_B(\infty)$ can be described in terms of such subtrees.

Definition 1. *We will call* X_k subtree *any subtree* T *of* $T_B(\infty)$ *which is rooted at an element* $a = (\underbrace{i, \dots, i}_{k}, a_{k+1}, \dots)$ *with* $a_{k+1} \leq i-1$ *and which is either the whole*

Algorithm 1 Efficient computation of the partitions of an integer.

```
Input: An integer n
Output: The partitions of n
begin
    Resu ← ∅;
    CurrentLevel ← {()};
    OldLevel ← ∅;
    l ← 0;
    while CurrentLevel ≠ ∅ do
        for each e in CurrentLevel do
            Compute p such that p_i = e_{i−1} for all i > 1 and p_1 = n − l;
            Add p to Resu;
        OldLevel ← CurrentLevel;
        CurrentLevel ← ∅;
        l ← l + 1;
        for each p in OldLevel do
            Add p^{↓1} to CurrentLevel;
            if p begins with a slippery plateau of length l then
                Add p^{↓l+1} to CurrentLevel;
        for each p in CurrentLevel do
            if n − l < p_1 then
                Remove p from CurrentLevel;
    Return(Resu);
end
```

subtree of $T_B(\infty)$ rooted at a if a has only one son, or a and its left subtree if a has two sons. Moreover, we define X_0 as a simple node.

The next proposition shows that all the X_k subtrees are isomorphic.

Proposition 6. *A X_k subtree, with $k \geq 1$, is composed by a chain of $k+1$ nodes (the rightmost chain) whose edges are labelled $1, 2, \dots, k$ and whose i-th node is the root of a X_{i-1} subtree for all i between 1 and $k+1$. See Figure 6.*

Proof. The claim is obvious for $k = 1$. Indeed, in this case the root a has the form $(i, a_2, \dots)$ with $a_2 \leq i-1$, therefore its left son has the form $(i+1, i-1, \dots)$, *i.e.* it starts with a cliff, and has only one son. This son also starts with a cliff; we can then deduce that X_1 is simply a chain, which is the claim for $k = 1$.

Suppose now the claim proved for any $i < k$ and consider the root a of a X_k subtree: $a = (\underleftrightarrow{i, \dots, i}_{k}, a_{k+1}, \dots)$ with $a_{k+1} \leq i-1$. Its left son is $a^{\downarrow 1} = (i+1, i, \dots, i, a_{k+1}, \dots)$ with $a_{k+1} \leq i-1$, therefore it is the root of a X_1 subtree. Moreover, $a^{\downarrow 1}$ has one right son: $a^{\downarrow_1\downarrow_2} = (i+1, i+1, i, \dots, i, a_{k+1}, \dots)$, which by definition is the root of a X_2 subtree. After $k-1$ such stages, we obtain $a^{\downarrow_1\downarrow_2\cdots\downarrow_{k-1}}$, which is equal to $(i+1, \dots, i+1, i, a_{k+1})$. This node is the root of a X_{k-1} subtree and has a right son: $a^{\downarrow_1\downarrow_2\cdots\downarrow_{k-1}\downarrow_k}$, *i.e.* $(\underbrace{i+1, \dots, i+1}_{k}, a_{k+1}, \dots)$, and we still have $a_{k+1} \leq i-1$. Therefore, this node is the root of a X_k subtree, and from the definition of $T_B(\infty)$ we know that it has no other son. This terminates the proof. □

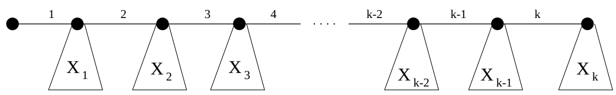

FIGURE 6. Self-referencing structure of X_k subtrees

This recursive structure and the above propositions allow us to give a compact representation of the tree by a chain:

Theorem 7. *The tree $T_B(\infty)$ can be represented by the infinite chain defined as follows: the i-th node of this chain, $(\underbrace{1, \dots, 1}_{i-1})$, is linked to the following node in the chain by an edge labelled with i and is the root of a X_{i-1} subtree. See Figure 7.*

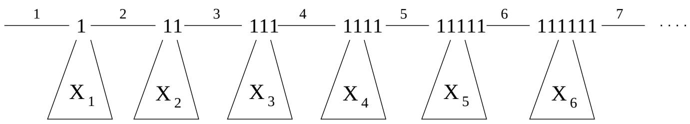

FIGURE 7. Representation of $T_B(\infty)$ as a chain

Moreover, we can prove a stronger property of each subtree in this chain:

Theorem 8. *The X_k subtree of $T_B(\infty)$ with root $(1, \dots, 1)$ contains exactly the partitions of length k.*

Proof. Because of their recursive structure shown in Proposition 6, X_k subtrees contain no edge with label greater than k. Therefore, if the root of a X_k subtree is of length k then all its nodes have length k. Moreover, no X_l subtrees with $l \neq k$ and with a root of length l can contain any node of length k. This remark, together with Theorem 7, implies the result. □

We can now state our last result:

Theorem 9. *Let $c(l,k)$ denote the number of paths in a X_k tree originating from the root and having length l. We have:*

$$c(l,k) = \begin{cases} 1 & \textit{if } l = 0 \textit{ or } k = 1 \\ \sum_{i=1}^{inf(l,k)} c(l-i, i) & \textit{otherwise} \end{cases}$$

Moreover, $|L_B(n)| = c(n,n)$ and the number of partitions of n with length exactly k is $c(n-k,k)$.

Proof. The formula for $c(l,k)$ is derived directly from the structure of X_k trees (Proposition 6 and Figure 6). To obtain $|L_B(n)|$, just remark that it immediately comes from Proposition 6 and Theorem 7 that the two subtrees obtained respectively from $T_B(\infty)$ and X_n by keeping only the nodes of depth at most n are isomorphic. The last formula is directly derived from Theorems 7 and 8. □

5. Perspectives

The self-similarity that appears during the construction of the lattices of integer partitions may be much more general, and should be compared with the notion of *duplications* in lattices [Day92]. This could lead to the definition of a new kind of duplications, and we would obtain this way the definition of a special class of lattices, which contains the lattices of integer partitions. Moreover, the ideas developped in this paper are very general and may be applied to other dynamical models, such as Chip Firing Games [BLS91], or tilings with flips [BNRR95, R99́]. It seems for example that the distributive lattice structure of the set of all the tilings of a figure with dominoes can be viewed as a consequence of the fact that it can be obtained from the set of all the tilings of a smaller figure by *duplication* of a part of this set.

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